JOURNAL OF APPROXIMATION THEORY **90**, 283–298 (1997)
ARTICLE NO. AT963081

Asymptotics of Diagonal Hermite–Padé Approximants to e^z

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Communicated by P. Borwein

Received October 13, 1995; accepted in revised form August 29, 1996

Let m be a fixed positive integer. We consider Hermite–Padé approximants to the exponential function

$$R(z)=\sum_{p=0}^{m} A_p(z)\, e^{pz}=O(z^{mn+n-1}),$$

where the degree of the polynomials A_p, $0\leqslant p\leqslant m$, is less than n. As $n\to\infty$, exact asymptotics for the A_p's and the remainder term R, along with an upper bound on the zeros of the polynomials A_p, are given. These asymptotics show that shifted Hermite–Padé approximants asymptotically minimize exponential polynomials of the above form on a disk $\{|z|\leqslant\rho\}$, provided ρ does not exceed π/m. These results generalize some of those obtained by Borwein (*Const. Approx.* **2** (1986), 291–302) on quadratic Hermite–Padé approximants. © 1997 Academic Press

1. INTRODUCTION

According to Mahler [12], there are two types of Hermite–Padé approximants. One of these types (the so-called German type or type II) consists of simultaneous rational approximants to a vector of functions. Considering such approximants to a vector of exponents of e, Hermite [9] proved in 1873 the transcendence of e. Ten years later, he introduced a second type of approximants (so-called Latin type or type I). For the vector of functions $(1, e^z, ..., e^{mz})$, these approximants consist of a set of polynomials $A_0, ..., A_m$, not all identically zero, of *degree less than* n such that

$$R(z):=\sum_{p=0}^{m} A_p(z)\, e^{pz}=O(z^{mn+n-1}). \tag{1.1}$$

Mahler [10] showed that they could also be used to prove the transcendence of e. These two types of Hermite–Padé approximants, explicitly

0021-9045/97 $25.00

constructed for various classes of functions, have many applications, in particular in number theory, where they provide measures of irrationality and transcendence proofs [11, 6]. The formal theory of Hermite–Padé approximants, initiated by Mahler [10], has been studied by many authors (see [2] for a bibliography).

If $m=1$ in (1.1) then we recover classical Padé approximation. The exponential function is one of the few examples where convergence results are known. This is Padé's theorem asserting that the polynomials A_0 and A_1, normalized so that $A_1(0)=1$, satisfy

$$A_0(z) \to -e^{z/2}, A_1(z) \to e^{-z/2} \qquad \text{as} \quad n \to \infty,$$

locally uniformly in $\mathbf{C}$. By means of explicit formulas, Borwein [4] established, among other results, a version for quadratic Hermite–Padé approximants ($m=2$), along with an asymptotic estimate for the remainder term $R(z)$.

It is the aim of this paper to extend Borwein's results to higher-dimensional approximations of the generality of (1.1). This we do by investigating integral representations of the $A_p(z)$, $p=0, ..., m$, and $R(z)$ in (1.1). The saddle point method is an efficient tool for such an analysis. It yields exact asymptotics, refining estimates obtained in [11] and [6]. By a different method, we obtain also for each degree an upper bound on the modulus of the zeros of the polynomials A_p, showing that, in this respect, they behave like zeros of the Padé approximants (see [17]).

The determination of the exact error estimate for rational best approximation to e^z on a finite interval (Meinardus conjecture) was achieved by Braess [5]. Applying the same method, Trefethen [18] settled the case of rational best approximation on a disk. In [4] and [8], similar results for quadratic approximation are given. By means of our asymptotics, we derive a generalization of these estimates to higher dimensions.

The case of non-diagonal quadratic Hermite–Padé approximants to e^z was recently studied by Driver [8] and the convergence of Hermite–Padé approximants of type II to exponential functions was investigated in [1].

2. ASYMPTOTICS OF HERMITE–PADÉ APPROXIMANTS TO EXPONENTIALS

The polynomials $A_0, ..., A_m$ in (1.1) are obtained by solving a linear system of $mn+n-1$ homogeneous equations with $mn+n$ unknown coefficients. Thus, non-trivial solutions always exist. As is well-known (cf. [11]), such a non-trivial solution can be given explicitly. Indeed, denoting by C_0 and C_∞ two circles, both centered at the origin and with radius less than 1 and greater than m, respectively, it can be checked that the expressions

$$A_p(z) = \frac{1}{2i\pi} \int_{C_0} \frac{e^{\zeta z} d\zeta}{\prod_{l=0}^{m} (\zeta + p - l)^n}, \qquad 0 \leqslant p \leqslant m, \tag{2.1}$$

$$R(z) = \frac{1}{2i\pi} \int_{C_\infty} \frac{e^{\zeta z}\, d\zeta}{\prod_{l=0}^{m} (\zeta - l)^n}, \tag{2.2}$$

satisfy (1.1) together with the prescribed hypothesis.

Since we shall consider in Section 3 the normalization obtained upon dividing (1.1) by the leading coefficient of A_m, we specify its value. Differentiating Eq. (2.1) $(n-1)$ times with $p = m$, we get as the leading coefficient of A_m

$$\frac{1}{2i\pi(n-1)!} \int_{C_0} \frac{d\zeta}{\zeta \prod_{l=1}^{m} (\zeta + l)^n},$$

whose value, by Cauchy's formula, is $(m!)^{-n}/(n-1)!$.

We study the asymptotics as $n \to \infty$ of the polynomials A_p, $0 \leqslant p \leqslant m$, by applying the saddle point method to the contour integrals given in (2.1). We recall the main results needed in the following and state them without proof in a form convenient for our purpose (cf. [7, 14]).

Theorem (Laplace's Method). *Let h and g be real continuous functions defined on some interval $[a, b]$. Let*

$$I_n = \int_a^b h(x)/g(x)^n \, dx.$$

Assume that g is positive and has an absolute minimum at an interior point x_0 in (a, b). Furthermore, assume that g' exists in some neighborhood of x_0, that $g''(x_0)$ exists, and that $g''(x_0) > 0$. Then, if $h(x_0) \neq 0$,

$$I_n = \sqrt{2\pi g(x_0)/g''(x_0)}\, \frac{h(x_0)}{\sqrt{n} g(x_0)^n} \left(1 + O\left(\frac{1}{n}\right)\right) \qquad \text{as} \quad n \to \infty.$$

Theorem (Saddle Point Method). *Let h and g be analytic functions in a simply connected open set Δ and assume that g has no zeros in Δ. Let Γ be a smooth oriented path with a finite length and endpoints a and b, lying in Δ. Moreover, let*

$$I_n = \int_\Gamma h(\zeta)/g(\zeta)^n \, d\zeta.$$

(i) *Assume that* $\min_{\zeta \in \Gamma} |g(\zeta)|$ *is attained at the endpoint* a *only and* $g'(a) \neq 0$. *Then, if* $h(a) \neq 0$,

$$I_n = \frac{h(a)}{ng'(a)\, g(a)^{n-1}} \left(1 + O\left(\frac{1}{n}\right)\right) \qquad \text{as} \quad n \to \infty.$$

(ii) *Assume that a point* ζ_0 *of* Γ, *different from an endpoint, is a non-degenerate critical point of* g (*i.e.*, $g'(\zeta_0) = 0$, $g''(\zeta_0) \neq 0$) *and let* ω *be the phase corresponding to the direction of the tangent to the oriented path at* ζ_0. *Suppose further that* $\min_{\zeta \in \Gamma} |g(\zeta)|$ *is attained at the point* ζ_0 *only. Then, if* $h(\zeta_0) \neq 0$,

$$I_n = \sqrt{2\pi g(\zeta_0)/g''(\zeta_0)}\, \frac{h(\zeta_0)}{\sqrt{n} g(\zeta_0)^n} \left(1 + O\left(\frac{1}{n}\right)\right), \tag{2.3}$$

as $n \to \infty$, *where the phase* ω_0 *of* $g''(\zeta_0)/g(\zeta_0)$ *is chosen to satisfy* $|\omega_0 + 2\omega| \leqslant \pi/2$. *Since* $g(\zeta) - g(\zeta_0) \sim (g''(\zeta_0)/2)(\zeta - \zeta_0)^2$ *as* $\zeta \to \zeta_0$ *along* Γ, *and* $|g(\zeta)/g(\zeta_0)| \geqslant 1$, *it is always possible to choose* ω_0 *uniquely in this way.*

Before applying these theorems, we introduce some notation. Following the usual Pochhammer convention, we set

$$(z)_m = z(z+1) \cdots (z+m-1), \qquad m > 0.$$

Consider the polynomial $(z)_{m+1}$ and let $\{\sigma_p = -p - \eta_p, 0 < \eta_p < 1\}_{p=0}^{m-1}$ be the set of its m critical points, that is, the m roots of its derivative. Let, for $0 \leqslant p \leqslant m-1$,

$$(\sigma_p)_{m+1} = (-1)^{p-1} \gamma_p, \gamma_p > 0, \qquad \text{and} \qquad \left.\frac{d^2(z)_{m+1}}{dz^2}\right|_{\sigma_p} = (-1)^p v_p, v_p > 0.$$

Note that the following identities hold

$$\eta_p = 1 - \eta_{m-p-1} \qquad \text{and} \qquad \gamma_p = \gamma_{m-p-1}, \qquad 0 \leqslant p \leqslant m-1,$$

so that in particular η_k equals 1/2 when m is an odd integer such that $m = 2k+1$. One can also easily check that

$$\gamma_{m-1} > \cdots > \gamma_{k+1} > \gamma_k < \gamma_{k-1} < \cdots < \gamma_0 \qquad \text{if} \quad m = 2k+1, \tag{2.4}$$

$$\gamma_{m-1} > \cdots > \gamma_{k+1} > \gamma_k = \gamma_{k-1} < \cdots < \gamma_0 \qquad \text{if} \quad m = 2k. \tag{2.5}$$

THEOREM 2.1. *Let*

$$\mu_{p,n} := \frac{(-1)^{pn}}{\sqrt{2\pi n v_p}\,\gamma_p^{n-1/2}}, \qquad 0 \leqslant p \leqslant m/2. \tag{2.6}$$

Then,

(i) *The following symmetry between Hermite–Padé approximants* $A_0, ..., A_m$ *to the exponential function, of degree less than n, holds:*

$$A_{m-p}(z) = (-1)^{mn+n-1} A_p(-z), \qquad 0 \leqslant p \leqslant m. \tag{2.7}$$

(ii) *Assume* $0 \leqslant p < m/2$. *Then, as* $n \to \infty$,

$$A_p(0) \sim (-1)^{mn} \mu_{p,n}. \tag{2.8}$$

Consequently, for n large, one can define $\tilde{A}_p$ (*resp.* $\tilde{A}_{m-p}$) *as the polynomial obtained upon dividing* A_p (*resp.* A_{m-p}) *by its nonzero constant coefficient. Then, as* $n \to \infty$,

$$\tilde{A}_p(z) \to e^{\eta_p z}, \tag{2.9}$$

locally uniformly in **C**. *In conjunction with* (2.7), *we get corresponding asymptotics* (2.8) *and* (2.9) *for* A_p, $m/2 < p \leqslant m$.

(iii) *If m is even, let* $A_{m/2}^{(1)}$ (*resp.* $A_{m/2}^{(2)}$) *be the subsequence of polynomials* $A_{m/2}$ *corresponding to even* (*resp. odd*) *indices n. Then,* $A_{m/2}^{(1)}$ *is an odd polynomial and* $A_{m/2}^{(2)}$ *is an even polynomial. Moreover, as* $n \to \infty$,

$$\frac{dA_{m/2}^{(1)}}{dz}(0) \sim 2\eta_{m/2}\mu_{m/2,n}, \qquad A_{m/2}^{(2)}(0) \sim 2\mu_{m/2,n}. \tag{2.10}$$

For n large, define $\tilde{A}_{m/2}^{(1)}$ (*resp.* $\tilde{A}_{m/2}^{(2)}$) *as the polynomial obtained upon dividing* $A_{m/2}^{(1)}$ (*resp.* $A_{m/2}^{(2)}$) *by its nonzero derivative at zero* (*resp. nonzero constant coefficient*). *Then, as* $n \to \infty$,

$$\tilde{A}_{m/2}^{(1)}(z) \to \frac{1}{2\eta_{m/2}}(e^{\eta_{m/2}z} - e^{-\eta_{m/2}z}), \qquad \tilde{A}_{m/2}^{(2)}(z) \to \frac{1}{2}(e^{\eta_{m/2}z} + e^{-\eta_{m/2}z}), \tag{2.11}$$

uniformly on compact subsets of **C**.

Proof. One immediately obtains relation (2.7) by changing ζ into $-\zeta$ in Eq. (2.1), which proves (i).

Next, to get assertion (ii), we apply the saddle point method to the integral representation (2.1) of the polynomial A_p. Here we have $g(\zeta) = (\zeta + p - m)_{m+1}$. We choose to deform the circle of integration C_0 to a rectangle R included in the region $\{-1<\mathrm{Re}(z)<1\}$ with vertices $(-a', -r)$, $(a, -r)$, (a, r), $(-a', r)$ where r is a positive real number large enough, say larger than $2\sqrt{m}$, and a and a' lie in (0,1). Then, the minimum of $|(\zeta+p-m)_{m+1}|$ on the vertical segment joining $(-a', r)$ to $(-a', -r)$ (resp., $(a, -r)$ to (a, r)) is attained at the point $-a'$ (resp. a) only. With the assumption made on r, the value of $|(\zeta+p-m)_{m+1}|$ on the two remaining horizontal segments is larger than both values at a and $-a'$. Indeed, this will be true if, for any $0\leqslant p\leqslant m$, $(p+1)^2\leqslant r^2+(p-1)^2$. In particular, it is satisfied when the inequality holds for $p=m$ (i.e., $4m\leqslant r^2$).

We first assume $p\neq 0$ (and $p<m/2$). Among all critical points of $(\zeta+p-m)_{m+1}$, two of them, $-\eta_{m-p}$ and $1-\eta_{m-p-1}=\eta_p$, lie in the segment $(-1, 1)$. We choose $a=\eta_p$ and $a'=\eta_{m-p}$ and decompose the rectangle R as $R_1\cup R_2$ where R_1 is the part of R lying in the left half-plane while R_2 is the part lying in the right half-plane. As $m-p>p$, we have $\gamma_p=\gamma_{m-p-1}<\gamma_{m-p}$. Thus, upon applying successively assertion (ii) of the saddle point method to the integral (2.1) evaluated on R_1 and R_2, we obtain that the contribution on R_1 is of order less than the contribution on R_2 and get

$$A_p(0)=\frac{(-1)^{(m-p)n}}{\sqrt{2\pi n v_p}\,\gamma_p^{n-1/2}}\left(1+O\left(\frac{1}{n}\right)\right), \tag{2.12}$$

$$A_p(z)=\frac{(-1)^{(m-p)n}\,e^{\eta_p z}}{\sqrt{2\pi n v_p}\,\gamma_p^{n-1/2}}\left(1+O\left(\frac{1}{n}\right)\right). \tag{2.13}$$

If $p=0$, the positive real number $1-\eta_{m-1}=\eta_0$ is the only critical point in $(-1, 1)$. We choose $a=\eta_0$ and some a' such that $|g(-a')|>\gamma_0$, which is possible. Applying respectively assertion (i) and (ii) of the saddle point method to the integral (2.1) on the segment from $(-a', 0)$ to $(-a', -r)$ and to the remaining part of R, we observe that the contribution on the segment is negligible with respect to the second contribution. Hence, equalities (2.12) and (2.13) extend to the case $p=0$. So far, we have proved (2.8) and the pointwise convergence of $\tilde{A}_p$ to the limit given in (2.9).

Let us proceed with assertion (iii) and assume $m=2p$ is even. Relation (2.7) implies that $A^{(1)}_{m/2}$ is an odd polynomial and $A^{(2)}_{m/2}$ is an even polynomial. We now establish (2.10). We have $\eta_{m-p}=\eta_p$. We choose $a=a'=\eta_p$ and let the reader check that the contributions of the arcs R_1 and R_2 for $(dA^{(1)}_{m/2}/dz)(0)$ (resp. $A^{(2)}_{m/2}(0)$) are of the same sign and same order, namely,

$$\frac{(-1)^{mn/2}\,\eta_{m/2}}{\sqrt{2\pi n v_{m/2}}\,\gamma_{m/2}^{n-1/2}}\left(1+O\left(\frac{1}{n}\right)\right)\qquad\left(\text{resp.}\ \frac{(-1)^{mn/2}}{\sqrt{2\pi n v_{m/2}}\,\gamma_{m/2}^{n-1/2}}\left(1+O\left(\frac{1}{n}\right)\right)\right).$$

We obtain in the same manner that the limits in (2.11) are pointwise satisfied.

It remains only to show in assertions (ii) and (iii) that the limits (2.9) and (2.11) are uniform with respect to z. To derive this, we prove that the normalized polynomials $\tilde{A}_p$, $\tilde{A}^{(1)}_{m/2}$, and $\tilde{A}^{(2)}_{m/2}$ are uniformly bounded. For z in a disk of radius ρ and ζ on the rectangle R, the modulus of $e^{\zeta z}$ is bounded, by $\kappa := e^{(1+2\sqrt{m})\rho}$, say. Together with (2.8), we get for n large,

$$|\tilde{A}_p(z)| \leqslant 2\,\frac{|A_p(z)|}{|\mu_{p,n}|} \leqslant \frac{\kappa}{\pi|\mu_{p,n}|} \int_a^b \frac{|\zeta'(t)|\,dt}{|\prod_{l=0}^m (\zeta(t)+p-l)|^n}, \tag{2.14}$$

assuming that the contour of integration is parameterized by a real number $t \in [a, b]$. Using Laplace's method to estimate the integral and supposing that ζ describes the same contours as previously, we obtain that

$$\begin{aligned} \int_a^b \frac{|\zeta'(t)|\,dt}{|\prod_{l=0}^m (\zeta(t)+p-l)|^n} &= \sqrt{\frac{-2\pi\gamma_p}{n\nu_p\zeta'(t_0)^2}}\;\frac{|\zeta'(t_0)|}{\gamma_p^n}\left(1+O\left(\frac{1}{n}\right)\right) \\ &= 2\pi\,|\mu_{p,n}|\left(1+O\left(\frac{1}{n}\right)\right), \end{aligned} \tag{2.15}$$

where t_0 is such that $\zeta(t_0)=\eta_p$. Thus by (2.14) and (2.15), the sequence of polynomials $\tilde{A}_p$ is locally uniformly bounded with n. This, together with Vitali's theorem, shows that the limit $\tilde{A}_p(z) \to e^{\eta_p z}$ is uniform on compact subsets of $\mathbf{C}$. The same argument can be applied to the two other sequences $\tilde{A}^{(1)}_{m/2}$ and $\tilde{A}^{(2)}_{m/2}$, when m is even. ∎

We proceed now with an analysis of the zeros of the polynomials A_p, $0 \leqslant p \leqslant m$. In view of (2.9) and (2.11) in Theorem 2.1, we know that, unless m is even and $p = m/2$, all the zeros of A_p tend to infinity. The next result gives for each degree an upper bound on their modulus.

THEOREM 2.2. *For any $m \geqslant 1$ and $n \geqslant 2$, all the zeros of the Hermite–Padé approximant $A_p(z)$ satisfying* (1.1) *lie in*

$$|z| \leqslant 2(n-1/3)\left[\sum_{k=1}^{p}\frac{1}{k} + \sum_{k=1}^{m-p}\frac{1}{k}\right], \qquad 0 \leqslant p \leqslant m, \tag{2.16}$$

where it is understood, in case $p=0$ or $p=m$, that the sum in (2.16) *ranging from $k=1$ to 0 vanishes.*

Remark. This theorem extends to Hermite–Padé approximants the known property that the zeros of the classical Padé approximants to e^z tend to infinity like $O(n)$. If m is even and $p = m/2$, the zeros of the limit functions in (2.11) are $\{\pm ki\pi/\eta_{m/2},\ k=0, 1, ...\}$ and $\{\pm(k+1/2)\,i\pi/\eta_{m/2},$

$k = 0, 1, ...\}$ respectively. Thus, zeros of $A_{m/2}$ converge to points in these sets or go to infinity.

Proof. We first derive a new expression for the polynomials A_p, $0 \leqslant p \leqslant m$. We shall need the differential operators $T_k := kI + D$, $-m \leqslant k \leqslant m$, where I denotes the identity and D the differentiation d/dz. Note that any two such operators commute. Differentiating Eq. (1.1) n times, we deduce that

$$e^{mz} T_m^n(A_m(z)) + \cdots + e^z T_1^n(A_1(z)) = O(z^{mn-1}).$$

Assume we want to compute A_p for some given value of p. Upon alternatively dividing by the nonvanishing function e^z and differentiating n times, we obtain the equality

$$e^{(m-p)z} T_{m-p+1}^n \cdots T_m^n(A_m(z)) + \cdots + T_1^n \cdots T_p^n(A_p(z)) = O(z^{n(m-p+1)-1}).$$

Dividing again this last equality by $e^{(m-p)z}$ and alternating n differentiations with division by e^z, we finally end up with

$$T_{-1}^n \cdots T_{-(m-p)}^n T_1^n \cdots T_p^n(A_p(z)) = O(z^{n-1}).$$

As $\deg A_p \leqslant n-1$, we deduce that

$$T_{-1}^n \cdots T_{-(m-p)}^n T_1^n \cdots T_p^n(A_p) = c_p z^{n-1},$$

where c_p is some nonzero constant. By inverting the operators T_p, we obtain the following expression for A_p,

$$A_p(z) = c_p T_p^{-n} \cdots T_1^{-n} T_{-(m-p)}^{-n} \cdots T_{-1}^{-n}(z^{n-1}). \tag{2.17}$$

Let S_{n-1} be any polynomial of degree $n-1$. The next step is the computation of $T_k^{-n}(S_{n-1})$, $-m \leqslant k \leqslant m$, $k \neq 0$. Writing the Taylor expansion

$$\left(1 + \frac{x}{k}\right)^{-n} = \sum_{j=0}^{\infty} (-1)^j \binom{n-1+j}{n-1} \left(\frac{x}{k}\right)^j,$$

we get

$$T_k^{-n}(S_{n-1}) = k^{-n} \sum_{j=0}^{n-1} (-1)^j \binom{n-1+j}{n-1} \frac{S_{n-1}^{(j)}}{k^j}, \tag{2.18}$$

where we use the fact that S_{n-1} is a polynomial of degree $n-1$. Here, we introduce the Padé approximant p_{n-1}/q_{n-1}, q_{n-1} monic, of type $(n-1,$

$n-1$) (i.e., $\deg p_{n-1} = \deg q_{n-1} = n-1$) to e^z which, by definition, is the unique rational function satisfying

$$q_{n-1}(z)\, e^z - p_{n-1}(z) = O(z^{2n-1}), \qquad \text{as} \quad z \to 0.$$

Thus, $q_{n-1}(z) = T_1^{-n}(z^{n-1})$. Plugging $k=1$, $S_{n-1} = z^{n-1}$ in (2.18) yields

$$q_{n-1}(z) = \sum_{j=0}^{n-1} (-1)^j \binom{n-1+j}{n-1} \frac{(n-1)!}{(n-1-j)!}\, z^{n-1-j}. \tag{2.19}$$

We now use the same method as in [3], appealing to the following theorem of Walsh (cf. [13, Theorem 18.1, p. 81])

(THEOREM (Walsh). *Let*

$$f(z) = \sum_{j=0}^{n} a_j z^j, \qquad g(z) = \sum_{j=0}^{n} b_j z^j = b_n \prod_{j=1}^{n} (z - \beta_j),$$

and

$$h(z) = \sum_{j=0}^{n} (n-j)!\, b_{n-j} f^{(j)}(z).$$

If all the zeros of $f(z)$ lie in a circular region A, then all the zeros of $h(z)$ lie in the point set C consisting of n circular regions obtained by translating A in the amount and direction of the vectors β_j.),

along with a result of Saff and Varga (cf. the upper bound in [17, Theorem 2.2, p. 198])

(THEOREM (Saff–Varga). *For any $m \geqslant 1$ and $n \geqslant 0$, all the zeros of the Padé approximant $p_{m,n}(z)/q_{m,n}(z)$ to e^z with* $\deg p_{m,n} = m$, $\deg q_{m,n} = n$, *lie in $\{|z| \leqslant m+n+4/3\}$*.)

Because of the identity $q_{m,n}(z) = p_{n,m}(-z)$, the previous inequality also holds for the zeros of $q_{m,n}(z)$ with $m \geqslant 0$ and $n \geqslant 1$.

Assume that all roots of S_{n-1} have modulus smaller than some real positive number ρ and apply Walsh's theorem with $f = S_{n-1}$ and $h = T_k^{-n}(S_{n-1})$. By (2.18), we must take in the theorem

$$g(z) = \sum_{j=0}^{n-1} b_{n-1-j} z^{n-1-j} = k^{-n} \sum_{j=0}^{n-1} \frac{(-1)^j}{(n-1-j)!} \binom{n-1+j}{n-1} \frac{z^{n-1-j}}{k^j}$$
$$= \frac{k^{-2n+1}}{(n-1)!}\, q_{n-1}(kz)$$

(see (2.19) for the last equality). From the Saff and Varga estimate, we know that the zeros of $q_{n-1}(kz)$ have modulus smaller than $2(n-1/3)/|k|$. Thus, zeros of $T_k^{-n}(S_{n-1})$ have modulus smaller than $\rho+2(n-1/3)/|k|$ by Walsh's theorem. In view of (2.17), we use this last assertion successively with $S_{n-1}=z^{n-1}$, $k=-1$, then $S_{n-1}=T_{-1}^{-n}(z^{n-1})$, $k=-2$, and so on, up to $S_{n-1}=T_{p-1}^{-n}\cdots T_1^{-n}T_{-(m-p)}^{-n}\cdots T_{-1}^{-n}(z^{n-1})$, $k=p$. The asserted upper bound (2.16) on the zeros of A_p follows. ∎

3. ASYMPTOTICS OF THE REMAINDER TERM $R(z)$

THEOREM 3.1. *As $n\to\infty$,*

$$R(z)\sim\frac{z^{mn+n-1}e^{mz/2}}{(mn+n-1)!},\tag{3.1}$$

uniformly on compact subsets of $\mathbf{C}$.

For a positive real number z, we note that (3.1) is consistent with the inequalities (3.8) of [6].

Proof. We use the integral representation (2.2) of $R(z)$. Since all critical points of the product in the denominator lie in the segment $(0, m)$ and the contour of integration has to enclose all of the points $0, ..., m$, it seems difficult to apply the saddle point method directly to the integral (2.2). So, we set $z=nw$, $w\in\mathbf{C}$, and begin with the study of the critical points of $g(\zeta)=e^{-w\zeta}(\zeta-m)_{m+1}$ or equivalently $\tilde{g}(\zeta)=e^{-w\zeta}(\zeta)_{m+1}$ after the substitution $\zeta\to\zeta+m$. Differentiating $\tilde{g}$, we obtain as critical point equation

$$w(\zeta)_{m+1}=(\zeta)'_{m+1}.\tag{3.2}$$

As the contour C_∞ has to encompass all points $0, ..., m$ or $-m, ..., 0$ after the change of variable $\zeta\to\zeta+m$, we look for a critical point of large modulus, say $|\zeta|\geqslant\kappa m$, for some real number $\kappa>1$. In this case, one can check easily from (3.2) that the solution satisfies $1/\zeta=O(w)$. Thus, by expanding (3.2) and dividing by ζ^m, we compute an expansion of the critical point $\tilde{\zeta}_0$ of $\tilde{g}$ in powers of w, which is of particular interest in regard to the normalization $z=nw$. One obtains $\tilde{\zeta}_0=(m+1)/w-m/2+O(w)$. Observe that $\tilde{\zeta}_0$ tends to infinity as w becomes small. Thus, as was assumed, $|\tilde{\zeta}_0|/m$ is larger than 1 and also $\tilde{\zeta}_0$ is nondegenerate for w small. Indeed, as $w\to 0$, the m other roots of (3.2) tend to those of $(\zeta)'_{m+1}$, which lie in $(-m, 0)$. The corresponding critical point ζ_0 of g satisfies

$$\zeta_0=\tilde{\zeta}_0+m=\frac{m+1}{w}+\frac{m}{2}+O(w).\tag{3.3}$$

Then, we have to specify a contour C_∞ through ζ_0, encompassing all points 0, ..., m and such that the modulus of g on C_∞ attains its minimum at ζ_0 only. To achieve this, we consider the level curves of $(\zeta-m)_{m+1}$ and $e^{-w\zeta}$ at ζ_0. The first one is a lemniscate $\mathscr{L}$ which encloses all roots of $(\zeta-m)_{m+1}$, if w is small. The second one is a line $\mathscr{D}$ of direction i/w tangent to $\mathscr{L}$ at ζ_0, since ζ_0 is critical. In the half-plane $\mathscr{H}$ delimited by $\mathscr{D}$ and containing $\mathscr{L}$, the modulus of $e^{-w\zeta}$ is larger than that of $e^{-w\zeta_0}$, while the modulus of $(\zeta-m)_{m+1}$ at the exterior of $\mathscr{L}$ is larger than that of $(\zeta_0-m)_{m+1}$. We construct C_∞ as follows. We take a segment on $\mathscr{D}$, containing ζ_0, and join its endpoints by a path in $\mathscr{H}$ surrounding $\mathscr{L}$. It is easily seen that such a C_∞ fulfills the desired requirements. Assertion (ii) of the saddle point method can thus be applied and we start with an estimation of $g(\zeta_0)=e^{-w\zeta_0}(\zeta_0-m)_{m+1}=e^{-w\zeta_0}(\tilde{\zeta}_0)_{m+1}$. From the recurrence formula for the Gamma function Γ, $\Gamma(z+1)=z\Gamma(z)$, we know that

$$g(\zeta_0)=e^{-w\zeta_0}\Gamma(\tilde{\zeta}_0+m+1)/\Gamma(\tilde{\zeta}_0).$$

In order to avoid the poles of Γ, we first assume that w does not lie on $(-\infty, 0]$ so that neither $\tilde{\zeta}_0$ nor ζ_0 are nonpositive real numbers. Using the well-known Stirling formula for Γ (see [14, p. 294]) and substituting ζ_0 and $\tilde{\zeta}_0$ by their expansion (3.3), one checks that

$$g(\zeta_0)=e^{-wm/2}\left(\left(\frac{m+1}{we^2}\right)\left(1+w+\frac{mw^2}{2(m+1)}\right)^{(1/w+1/2)}\right.$$
$$\left.\times\left(1-\frac{mw}{2(m+1)}\right)\right)^{m+1}(1+O(w^2)). \tag{3.4}$$

Next, we compute $g''(\zeta_0)/g(\zeta_0)$. Introducing the logarithmic derivative of Γ or digamma function ψ (cf. [14] for a definition and some properties), we get

$$g''(\zeta_0)/g(\zeta_0)=\psi'(\zeta_0+1)-\psi'(\zeta_0-m).$$

For z not in $(-\infty, 0]$, the approximation $\psi'(z)=1/z+1/2z^2+O(1/z^3)$ holds. Thus, with (3.3), one obtains

$$g''(\zeta_0)/g(\zeta_0)=-\frac{w^2}{m+1}(1+O(w)). \tag{3.5}$$

Replacing w by z/n in (3.4) and (3.5) and evaluating the expression (2.3) with the correct choice for the phase of $g''(\zeta_0)/g(\zeta_0)$ leads to the following estimate for $z\in\mathbf{C}\backslash(-\infty, 0]$,

$$R(z)\sim\sqrt{\frac{(m+1)n}{2\pi}}\left(\frac{e}{(m+1)\,n}\right)^{(m+1)\,n} z^{mn+n-1}e^{mz/2}(1+\eta(z)),$$

where the function $1+\eta(z)$ comes from the factor $(1+O(1/n))$ in (2.3). By an inspection of Laplace's method for contour integrals (cf. [14, Chapter 4, Theorem 6.1]), one shows in our case that the factor $(1+O(1/n))$ is still of order 1, when w remains in a neighborhood of the origin, in particular when substituting z/n for w and $n\to\infty$. In terms of factorials, we get

$$R(z)\sim\frac{z^{mn+n-1}e^{mz/2}}{(mn+n-1)!},$$

as $n\to\infty$. To obtain the previous asymptotics for any complex number z, it suffices to observe from (2.2) that the following relation holds

$$R(-z)=(-1)^{mn+n-1}\,e^{-mz}R(z),\qquad z\in\mathbf{C}. \tag{3.6}$$

The uniformity of the asymptotic follows from Vitali's theorem and the fact that the quotient $(mn+n-1)!\,e^{-mz/2}R(z)/z^{mn+n-1}$ is uniformly bounded on compact subsets of $\mathbf{C}$. Indeed, we infer from (2.2) that

$$|R(z)|\leqslant\frac{1}{2\pi}\int_a^b\frac{e^{\mathrm{Re}(\zeta(t)\,z)}\,|\zeta'(t)|\,dt}{|\prod_{l=0}^{m}(\zeta(t)-l)^n|},$$

where the contour of integration is parameterized by a real number $t\in[a,b]$. Using Laplace's method to estimate the integral and keeping the contour unchanged, we get in the same manner as above that

$$|R(z)|\leqslant\frac{|z|^{mn+n-1}\,e^{m\,\mathrm{Re}(z)/2}}{(mn+n-1)!}\left(1+O\left(\frac{1}{n}\right)\right),\qquad z\in\mathbf{C}\backslash(-\infty,0],$$

where the factor $(1+O(1/n))$ depends on z or $w=z/n$. As z stays in some compact subset of $\mathbf{C}$, w tends to zero, and so remains in a neighborhood of the origin. Then by inspecting Laplace's method as previously, one can make the factor $(1+O(1/n))$ independent of w or z and still of order 1 as $n\to\infty$. Finally, by (3.6), this inequality is in fact satisfied for any $z\in\mathbf{C}$, thereby proving our contention. ∎

Let us now illustrate Theorems 2.1 and 3.1 by considering two examples, namely the classical Padé approximant $(m=1)$ and the quadratic one $(m=2)$.

In case $m=1$, we consider critical points of $z(z+1)$. Using the notations introduced before Theorem 2.1, we have $\{\sigma_0=-1/2,\ \gamma_0=1/4,\ v_0=2,\ \eta_0=1/2\}$. Thus, from Theorem 2.1, we get for the two polynomials A_0 and A_1 of degree $n-1$,

$$A_0(z)\sim(-1)^n\,4^{(n-1)}e^{z/2}/\sqrt{\pi n}\qquad\text{and}\qquad A_1(z)\sim(-1)^{n-1}\,4^{(n-1)}e^{-z/2}/\sqrt{\pi n},$$

as $n \to \infty$. If we consider normalized Padé approximants obtained upon division by the constant coefficient $A_1(0)$, we recover the classical Padé theorem (see [15]). Further, Theorem 3.1 gives the error estimate in diagonal Padé approximation (i.e., $\deg A_0 = \deg A_1 = n-1$), namely

$$e^z + A_0(z)/A_1(z) \sim (-1)^{n-1} \frac{\sqrt{\pi n}}{4^{n-1}} \frac{z^{2n-1} e^z}{(2n-1)!},$$

which, by Stirling's formula, can be rewritten as

$$e^z + A_0(z)/A_1(z) \sim (-1)^{n-1} \frac{(n-1)!\,(n-1)!}{(2n-2)!\,(2n-1)!}\, z^{2n-1} e^z$$

(compare [5, Eq. (8)]).

In case $m=2$, we compute critical points of $z(z+1)(z+2)$, obtaining $\{\sigma_0 = -1 + 1/\sqrt{3},\ \gamma_0 = 2/(3\sqrt{3}),\ \nu_0 = 2\sqrt{3},\ \eta_0 = 1 - 1/\sqrt{3}\}$ and $\{\sigma_1 = -1 - 1/\sqrt{3}, \gamma_1 = 2/(3\sqrt{3}), \nu_1 = 2\sqrt{3}, \eta_1 = 1/\sqrt{3}\}$. Moreover,

$$\mu_{0,n} = \frac{1}{3\sqrt{2n\pi}} \left(\frac{3\sqrt{3}}{2}\right)^n, \qquad \mu_{1,n} = (-1)^n \mu_{0,n}.$$

Thus,

$$\begin{aligned}
A_2(z) &\sim (-1)^{n-1} \mu_{0,n} e^{-(1-1/\sqrt{3})z}, \\
A_1(z) &\sim (-1)^n \mu_{0,n} (e^{z/\sqrt{3}} + (-1)^{n-1} e^{-z/\sqrt{3}}), \\
A_0(z) &\sim \mu_{0,n} e^{(1-1/\sqrt{3})z}.
\end{aligned}$$

Keeping in mind the normalization chosen in [4] and changing z into $-z$, one checks that the three asymptotics above agree with those of Proposition 3 of [4]. On the other hand, we have from Theorem 3.1 that the remainder $R(z)$ verifies

$$R(z) \sim \frac{z^{3n-1} e^z}{(3n-1)!}.$$

Considering Hermite–Padé approximants normalized by the leading coefficient of A_2, whose value by the computation preceding the statement of Laplace's method in Section 2 is $2^{-n}/(n-1)!$, we get for the normalized remainder the asymptotic

$$\frac{2^n (n-1)!\, z^{3n-1} e^z}{(3n-1)!},$$

in agreement with Proposition 2 of [4].

In case $m > 2$, we observe, in view of (2.4) and (2.5) along with (2.6), that the dominant terms in the sum $\sum_{p=0}^{m} A_p(z)\, e^{pz}$ are the middle terms, namely,

$$\begin{aligned} &A_{k+1}(z)\, e^{(k+1)z} + A_k(z)\, e^{kz} \\ &\qquad = [A_{k+1}(z)\, e^{z/2} + A_k(z)\, e^{-z/2}]\, e^{mz/2}, \qquad \text{if} \quad m = 2k+1, \end{aligned}$$

and

$$\begin{aligned} &A_{k+1}(z)\, e^{(k+1)z} + A_k(z)\, e^{kz} + A_{k-1}(z)\, e^{(k-1)z} \\ &\qquad = [A_{k+1}(z)\, e^{z} + A_k(z) + A_{k-1}(z)\, e^{-z}]\, e^{mz/2}, \qquad \text{if} \quad m = 2k. \end{aligned}$$

Replacing the polynomials A_{k-1}, A_k, A_{k+1} by their asymptotics in Theorem 2.1, one obtains that the bracketed expressions vanish, which is consistent with Theorem 3.1.

Next, we consider the following minimization problem,

Problem. Given a value of n, minimize uniformly on the disk $\{z \in \mathbf{C}, |z| \leqslant \rho\}$,

$$r_n(z) = \sum_{p=0}^{m} a_{p,n}(z)\, e^{pz},$$

where the coefficients $a_{p,n}$ are polynomials of degree less than or equal to n, and $a_{m,n}$ has leading coefficient 1.

An asymptotic estimate for the norm of solutions to this problem was established by Borwein [4] in the case $\rho = 1$ and $m = 2$. We show that Theorem 3.1 allows one to extend this estimate to any value of m as far as the value of ρ does not exceed π/m.

Following the ideas of Braess [5], we consider shifted Hermite–Padé approximants in degree n. Let

$$a_{p,n}^*(z) = n!\,(m!)^{n+1} A_p\left(z - \frac{m\rho^2}{2(mn+m+n)}\right), \qquad 0 \leqslant p \leqslant m,$$

$$r_n^*(z) = n!\,(m!)^{n+1} R\left(z - \frac{m\rho^2}{2(mn+m+n)}\right),$$

where the factor $n!\,(m!)^{n+1}$ in the above formulas comes from the normalization by the leading coefficient of A_m, so that $a_{m,n}^*$ has leading coefficient 1.

Denoting by $\|\cdot\|_\rho$, the supremum norm on $\{z \in \mathbf{C}, |z| \leqslant \rho\}$, we have

THEOREM 3.2. (i)

$$\left\| \sum_{p=0}^{m} a_{p,n}^*(z)\, e^{pz} \right\|_\rho \sim \frac{n!(m!)^{n+1}}{(mn+m+n)!}\, \rho^{mn+m+n}.$$

(ii) *Assume* $\rho < \pi/m$ *and let*

$$w_n^* = \min_{\substack{\deg a_{p,n} \leqslant n,\ 0 \leqslant p \leqslant m \\ a_{m,n} = z^n + \cdots}} \left\| \sum_{p=0}^{m} a_{p,n}(z)\, e^{pz} \right\|_\rho.$$

Then

$$w_n^* \sim \frac{n!\,(m!)^{n+1}}{(mn+m+n)!}\, \rho^{mn+m+n}.$$

Proof. From Theorem 3.1, we have for $|z| = \rho$,

$$|r_n^*(z)| \sim \frac{n!\,(m!)^{n+1}}{(mn+m+n)!}\, \rho^{mn+m+n},$$

since

$$\left(z - \frac{m\rho^2}{2(mn+m+n)}\right)^{mn+m+n} \sim z^{mn+m+n} e^{-m\rho^2/2z}.$$

This shows (i).

The proof of (ii) goes as in [4]. Assume that there exist polynomials $\tilde{a}_{p,n}$, $0 \leqslant p \leqslant m$, with $\deg \tilde{a}_{p,n} \leqslant n$ and $\tilde{a}_{m,n}$ having leading coefficient 1 such that for $|z| = \rho$,

$$\left| \sum_{p=0}^{m} \tilde{a}_{p,n}(z)\, e^{pz} \right| < \left| \sum_{p=0}^{m} a_{p,n}^*(z)\, e^{pz} \right|.$$

Then, by Rouché's theorem, the difference

$$\sum_{p=0}^{m} (\tilde{a}_{p,n} - a_{p,n}^*)\, e^{pz} \tag{3.7}$$

has at least $mn+m+n$ zeros in $\{|z| \leqslant \rho\}$ for n large. But this contradicts the fact that a nonzero expression $\sum_{p=0}^{m} a_{p,n}(z)\, e^{pz}$ with $\sum_{p=0}^{m} \deg a_{p,n} = h$ can have at most $h + m + m\rho/\pi$ zeros in the disk $\{|z| \leqslant \rho\}$ (cf. [16, Problem 206.2]). Indeed, the sum of the degrees of the polynomials $\tilde{a}_{p,n} - a_{p,n}^*$ is at most $mn + n - 1$. Thus, expression (3.7) can have at most $mn+m+n-1$ zeros in $\{|z| \leqslant \rho\}$ since we are assuming $m\rho/\pi < 1$. ∎

REFERENCES

1. A. I. Aptekarev, Convergence of rational approximations to a set of exponents, *Moscow Univ. Math. Bull.* **36** (1981), 81–86.
2. A. I. Aptekarev and H. Stahl, Asymptotics of Hermite–Padé polynomials, *in* "Progress in Approximation Theory" (A. A. Gonchar and E. B. Saff, Eds.), pp. 127–167, Springer-Verlag, New York/Berlin, 1992.
3. L. Baratchart, E. B. Saff, and F. Wielonsky, Rational interpolation of the exponential function, *Canad. J. Math.* **47** (1995), 1121–1147.
4. P. B. Borwein, Quadratic Hermite–Padé approximation to the exponential function, *Const. Approx.* **2** (1986), 291–302.
5. D. Braess, On the conjecture of Meinardus on rational approximation of e^x, II, *J. Approx. Theory* **40** (1984), 375–379.
6. G. V. Chudnovsky, Hermite–Padé approximations to exponential functions and elementary estimates of the measure of irrationality of π, *in* "Lecture Notes in Math.," Vol. 925, pp. 299–322, Springer-Verlag, New York, 1982.
7. N. G. De Bruijn, "Asymptotic Methods in Analysis," 2nd ed., Wiley, New York, 1961.
8. K. A. Driver, Non-diagonal quadratic Hermite–Padé approximation to the exponential function, *J. Comp. Appl. Math.* **65** (1995), 125–134.
9. C. Hermite, Sur la fonction exponentielle, *C. R. Acad. Sci. Paris* **77** (1873), 18–24, 74–79, 226–233, 285–293.
10. K. Mahler, Zur Approximation der Exponentialfunktion und des Logarithmus, I, II, *J. Reine Angew. Math.* **166** (1931), 118–137, 138–150.
11. K. Mahler, Applications of some formulas by Hermite to the approximation of exponentials and logarithms, *Math. Ann.* **168** (1967), 200–227.
12. K. Mahler, Perfect systems, *Comp. Math.* **19** (1968), 95–166.
13. M. Marden, "Geometry of Polynomials," Mathematical Surveys, Vol. 3, American Mathematical Society, Providence, RI, 1966.
14. F. Olver, "Asympotics and Special Functions," Academic Press, San Diego, 1974.
15. O. Perron, "Die Lehre von den Kettenbrüchen," Vol. 2, 3rd ed., Teubner, Stuttgart, 1957.
16. G. Pólya and G. Szegö, "Problems and Theorems in Analysis," Vol. I, Springer-Verlag, Berlin, 1972.
17. E. B. Saff and R. S. Varga, On the zeros and poles of Padé approximations to e^z, II, *in* "Padé and Rational Approximations: Theory and Applications" (E. B. Saff and R. S. Varga, Eds.), pp. 195–213, Academic Press, New York, 1977.
18. L. N. Trefethen, The asymptotic accuracy of rational best approximations to e^z on a disk, *J. Approx. Theory* **40** (1984), 380–383.